Quarks in Coulomb gauge perturbation theory

C. Popovici, P. Watson, and H. Reinhardt

Institut für Theoretische Physik, Universität Tübingen,
Auf der Morgenstelle 14, D-72076 Tübingen, Deutschland

Coulomb gauge quantum chromodynamics within the first order functional formalism is considered. The quark contributions to the Dyson-Schwinger equations are derived and one-loop perturbative results for the two-point functions are presented.

1. INTRODUCTION

For more than thirty years, quantum chromodynamics [QCD] has been established as the theory of the strong interaction, yet fundamental questions still remain unanswered. Foremost amongst these are the issues of confinement and the details of the hadron spectrum. As was recognized early on, studies in Coulomb gauge (where the framework is naturally connected to physical degrees of freedom) should provide for a promising arena of endeavor. However, what was also recognized at the time was that the inherent noncovariance of Coulomb gauge made technical progress difficult [1]. This is one reason why the linear covariant gauges (such as Landau and Feynman gauges) have been preferred for both perturbative and nonperturbative calculations. Clearly though, since the physical world is gauge invariant, it is certainly worthwhile to study the different gauges to gain more insight. Perturbation theory is one area where results in different gauges can be unambiguously compared. In addition, perturbative results form the basis of many nonperturbative studies since they provide a reliable way of dealing with ultraviolet divergences.

In the last few years, Coulomb gauge studies have been enjoying significant progress. There exists an appealing picture for confinement: the Gribov–Zwanziger scenario [2, 3, 4] whereby the temporal component of the gluon propagator provides for a long-range confining force whilst the transverse spatial components are suppressed in the infrared (and therefore do not appear as asymptotic states). Recent lattice studies seem to confirm this [5]. A Hamiltonian-based approach [6, 7, 8, 9, 10] also exists and proves adept at describing various physical features of the system [11, 12, 13]. More pertinent to this study, results in the Lagrange-based (Dyson–Schwinger) functional integral approach (which is especially suitable for perturbative calculations) to Yang–Mills theory have recently become available [14, 15, 16, 17, 18].

In this work, we consider the quark contributions in the first order functional approach to Coulomb gauge QCD, using results obtained in the pure Yang–Mills theory as a basis [14, 15]. We derive the relevant Dyson–Schwinger equations (or their modification from the pure Yang–Mills theory) for the two-point functions and then consider their perturbative counterparts. Using techniques based on integration by parts and differential equations, we evaluate the noncovariant integrals and present the results for the one-loop, two-point dressing functions.

The paper is organized as follows. We begin in the next section by considering the first order formalism and deriving the relevant field equations of motion. The Feynman rules and general decompositions of the two-point functions are obtained in Section 3. In Section 4, the Dyson–Schwinger equations are presented in detail. The one-loop perturbative dressing functions are derived in Section 5. The evaluation of the massive noncovariant integrals arising in the one-loop calculations is presented in Section 6. The results for the two-point functions are presented in Section 7. A summary and an outlook are given in Section 8. Various technical details are discussed in the Appendices.

2. FUNCTIONAL FORMALISM

Throughout this work, we will use the notations and conventions introduced in [14, 15]. We work in Minkowski space (until the perturbative integrals are to be explicitly evaluated whereupon we analytically continue to Euclidean space), with the metric $g_{\mu\nu} = \text{diag}(1, -\vec{1})$. Greek letters ($\mu, \nu...$) denote Lorentz indices, roman letters ($i, j...$) denote spatial indices and superscripts ($a, b, c...$) stand for color indices in the adjoint representation. Also, configuration space coordinates may be denoted with subscript ($x, y, z...$) when no confusion arises. The Dirac γ matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The Yang–Mills and quark actions are given by:

$$\mathcal{S}_{YM} = \int d^4x \left[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \right], \tag{2.1}$$

$$\mathcal{S}_q = \int d^4x \; \bar{q}(\imath\gamma^0 D_0 + \imath\gamma^j D_j - m)q \tag{2.2}$$

and the QCD action, $\mathcal{S}_{QCD} = \mathcal{S}_{YM} + \mathcal{S}_q$, is invariant under a local $SU(N_c)$ gauge transform. In the above quark action, q ($\bar{q}$) denotes the (conjugate) quark field, the Dirac and color indices in fundamental representation are implicit and we have N_f flavors of identical quarks and N_c colors. The notation γ^i refers to the spatial component of the Dirac γ matrices, where the minus sign arising from the metric has been explicitly extracted when appropriate. The temporal and spatial components of the covariant derivative (also implicitly in the fundamental color representation) are given by:

$$
\begin{aligned}
D^0 &= \partial^0 - \imath g T^c \sigma^c, \\
D_i &= \partial_i + \imath g T^c A_i^c,
\end{aligned}
\tag{2.3}
$$

where we also rename the temporal component of the gauge field (A^{0a}) to σ^a. The covariant derivative and (in the Yang–Mills action) the field strength tensor F are defined in terms of the gauge potential A:

$$
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c,
\tag{2.4}
$$

where the f^{abc} are the (totally antisymmetric) structure constants of the $SU(N_c)$ group, whose (hermitian) generators satisfy $[T^a, T^b] = \imath f^{abc} T^c$ and we use the normalization condition $\text{tr}(T^a T^b) = \delta^{ab}/2$. For later use, the color factor (Casimir invariant) for the gap equation will be written $C_F = (N_c^2 - 1)/2N_c$.

Consider now the functional integral:

$$
Z = \int \mathcal{D}\Phi \exp\{\imath \mathcal{S}_{QCD}\},
\tag{2.5}
$$

where $\mathcal{D}\Phi$ denotes the functional integration measure for all fields. Since the action is invariant under gauge transformations, Z is divergent by virtue of the integration over the gauge group. To overcome this problem we use the Faddeev-Popov technique and introduce a gauge-fixing term (for Coulomb gauge: $\vec{\nabla}\cdot\vec{A}^a = 0$) along with an associated ghost term in standard fashion. At this stage, we also convert to the first order formalism. This entails introducing three auxiliary fields: $\vec{\pi}$, ϕ and τ in order to linearize the action with respect to the temporal component of the gauge field (σ) in local fashion. Classically, $\vec{\pi}$ would be the momentum conjugate to $\vec{A}$ and is here transverse ($\vec{\nabla}\cdot\vec{\pi}^a = 0$); ϕ is a scalar field such that $-\vec{\nabla}\phi^a$ is the longitudinal component of the conjugate momentum field and τ is a trivial Lagrange multiplier field. All this is described in detail in Ref. [14] and is not repeated here. Indeed, in this work the details of the gauge-fixing and first order formalism are unimportant because the quarks do not directly couple to any of these fields (including the ghosts). What is however important later on is that these extra fields will formally enter the discussion of the Legendre transform (through partial functional derivatives) which, in principle, gives additional terms but which vanish at one-loop order perturbatively. Thus, the reader need only be aware of the existence of such fields and be assured that the specific details are not relevant to the present study.

The full generating functional of the theory is given by the functional integral, Eq. (2.5), in the presence of sources. Making the sources relevant to this work explicit and denoting the rest with dots, we have:

$$
Z[J] = \int \mathcal{D}\Phi \exp\left\{\imath \mathcal{S}_{QCD} + \imath \int d^4x (\bar{\chi}_x q_x + \bar{q}_x \chi_x + \rho^a \sigma^a + \vec{J}^a \cdot \vec{A}^a + \kappa^a \phi^a + \vec{K}^a \cdot \vec{\pi}^a + \ldots)\right\}.
\tag{2.6}
$$

The field equations of motion (from which the Dyson–Schwinger equations follow) are derived from the observation that the integral of a total derivative vanishes, up to possible boundary terms. We use the usual assumption that these boundary terms do not contribute [14]. For the quark field (we will return to other fields below), we have:

$$
\int \mathcal{D}\Phi \frac{\delta}{\delta \imath \bar{q}_x} \exp\left\{\imath \mathcal{S}_{YM} + \imath \int d^4x \left[\bar{q}_x(\imath\gamma^0 D_{0x} + \imath\gamma^j D_{jx} - m)q_x + \bar{\chi}_x q_x + \bar{q}_x \chi_x\right] + \ldots\right\} = 0
\tag{2.7}
$$

(again the dots represent those source terms that do not play a role here). Using the expression for the components of the covariant derivative, Eq. (2.3), it follows that

$$
\int \mathcal{D}\Phi [(\imath\gamma^0 \partial_{0x} + \imath\gamma^k \nabla_{kx} + g T^c \gamma^0 \sigma_x^c - g T^c \gamma^k A_{kx}^c - m)q_x + \chi_x] \exp\{\imath \mathcal{S}\} = 0,
\tag{2.8}
$$

where $\mathcal{S}$ is the full action plus source terms.

The generating functional, $Z[J]$, generates both connected and disconnected Green's functions. However, in practice we work with connected two-point and one-particle irreducible n-point Green's functions. The generating functional of connected Green's functions is $W[J]$, where $Z = e^W$. We introduce a bracket notation for the functional derivatives

of W, such that for a generic source denoted by J_α (the index denotes both the type and all other possible attributes of the field):

$$\frac{\delta W}{\delta \imath J_\alpha} = <\imath J_\alpha> . \tag{2.9}$$

Converting Eq. (2.8) into derivatives of $W[J]$ we obtain :

$$(\imath\gamma^0\partial_{0x} + \imath\gamma^k\nabla_{kx} - m) <\imath\bar{\chi}_x> + gT^c\left\{\gamma^0[<\imath\rho_x^c><\imath\bar{\chi}_x> + <\imath\rho_x^c\imath\bar{\chi}_x>] - \gamma^k[<\imath J_{kx}^c><\imath\bar{\chi}_x> + <\imath J_{kx}^c\imath\bar{\chi}_x>]\right\} + \chi_x = 0. \tag{2.10}$$

We define the generic classical field (we use the same notation for the classical fields and for the quantum fields which are integrated over since no confusion will arise) to be:

$$\Phi_\alpha = \frac{1}{Z}\int\mathcal{D}\Phi\,\Phi_\alpha\exp\{\imath\mathcal{S}\} = \frac{1}{Z}\frac{\delta Z}{\delta\imath J_\alpha}. \tag{2.11}$$

The generating functional of the proper (one-particle irreducible) Green's functions is the effective action, $\Gamma[\Phi]$, (which is a function of the classical fields) and is defined via the Legendre transform of $W[J]$:

$$\Gamma[\Phi] = W[J] - \imath\Phi_\alpha J_\alpha. \tag{2.12}$$

(We use the common convention that summation over all discrete indices and integration over continuous arguments is implicit). This gives:

$$<\imath J_\alpha> = \frac{\delta W}{\delta\imath J_\alpha} = \Phi_\alpha \quad \text{and} \quad <\imath\Phi_\alpha> = \frac{\delta\Gamma}{\delta\imath\Phi_\alpha} = -J_\alpha. \tag{2.13}$$

We have used the same bracket notation to denote derivatives of Γ with respect to fields – no confusion arises since we never mix derivatives with respect to sources and fields. Note that care must be taken to observe the correct minus signs associated with the Grassmann fields and sources. The equation of motion, Eq. (2.10), in terms of proper functions (and from which we will derive the quark gap equation) now reads:

$$<\imath\bar{q}_x> = -\imath(\imath\gamma^0\partial_{0x} + \imath\gamma^k\nabla_{kx} - m)\imath q_x + gT^c\gamma^0\left[\sigma_x^c q_x + <\imath\rho_x^c\imath\bar{\chi}_x>\right] - gT^c\gamma^k\left[A_{kx}^c q_x + <\imath J_{kx}^c\imath\bar{\chi}_x>\right]. \tag{2.14}$$

In a similar fashion, the quark contributions to the field equations of motion for $<\imath\sigma_x^a>$ and $<\imath A_{ix}^a>$ are given by (the rest of these equations are simply the Yang–Mills expressions derived previously in Ref. [14] and are not important here):

$$<\imath\sigma_x^a> = g\bar{q}_x T^a\gamma^0 q_x + g\text{Tr}\left\{T^a\gamma^0 <\imath\bar{\chi}_x\imath\chi_x>\right\} + \dots , \tag{2.15}$$

$$<\imath A_{ix}^a> = -g\bar{q}_x T^a\gamma^i q_x - g\text{Tr}\left\{T^a\gamma^i <\imath\bar{\chi}_x\imath\chi_x>\right\} + \dots , \tag{2.16}$$

where the trace is over Dirac and fundamental color indices.

3. FEYNMAN RULES AND DECOMPOSITIONS

Let us now discuss the Feynman rules and decompositions of the Green's functions. The tree-level quark propagator can be derived directly from the quark equation of motion in terms of connected functions, Eq. (2.10), by functionally differentiating and neglecting the interaction terms. For the noncovariant case here we obtain (after Fourier transforming to momentum space):

$$W_{\bar{q}q}^{(0)}(k) = -\imath\frac{\gamma^0 k_0 - \gamma^i k_i + m}{k_0^2 - \vec{k}^2 - m^2}. \tag{3.1}$$

Later on (where appropriate), we will use the usual notation $\not{k} = \gamma^0 k_0 - \gamma^i k_i$. The tree-level gluon propagators needed in this work have been derived in [14] and are given by:

$$W_{AAij}^{(0)}(k) = \frac{\imath t_{ij}(k)}{k_0^2 - \vec{k}^2}, \quad W_{\sigma\sigma}^{(0)}(k) = \frac{\imath}{\vec{k}^2} \tag{3.2}$$

where $t_{ij}(k) = \delta_{ij} - k_i k_j / \vec{k}^2$ is the transverse spatial projector. It is understood that the denominator factors involving both temporal and spatial components implicitly carry the Feynman prescription, i.e.,

$$\frac{1}{(k_0^2 - \vec{k}^2)} \rightarrow \frac{1}{(k_0^2 - \vec{k}^2 + \imath 0_+)}, \tag{3.3}$$

such that the analytic continuation to the Euclidean space can be performed.

The tree-level quark proper two-point function is derived from the quark equation of motion in terms of proper functions, Eq. (2.14):

$$\Gamma_{\bar{q}q}^{(0)}(k) = \imath (\slashed{k} - m). \tag{3.4}$$

There are two tree-level quark-gluon vertices, spatial and temporal, again obtained by taking the appropriate functional derivatives:

$$\begin{aligned} \Gamma_{\bar{q}q\sigma}^{(0)a} &= gT^a\gamma^0, \\ \Gamma_{\bar{q}qAj}^{(0)a} &= -gT^a\gamma^j. \end{aligned} \tag{3.5}$$

In addition to their tree-level forms, we will also require the general decompositions for the quark two-point functions (connected and proper) and the relationship between them. Because we work in a noncovariant setting, the usual arguments must be modified to include separately the temporal and spatial components. Starting with the quark propagator, we observe that

$$W_{\bar{q}q}(k^0, \vec{k}) \sim \int \mathcal{D}\Phi\, \bar{q}q \exp\{\imath \mathcal{S}\} \tag{3.6}$$

such that under both time-reversal and parity transforms, the propagator will remain unchanged – the bilinear combination of fields is scalar. Since the propagator depends on both k^0 and $\vec{k}$, it has thus *four* components in distinction to the covariant case where there are only two. We thus write

$$W_{\bar{q}q}(k) = -\frac{\imath}{k_0^2 - \vec{k}^2 - m^2} \left\{ k_0\gamma^0 F_t(k) - k_i\gamma^i F_s(k) + M(k) + k_0 k_i \gamma^0\gamma^i F_d(k) \right\} \tag{3.7}$$

where all dressing functions are functions of both k_0^2 and $\vec{k}^2$. At tree-level, $F_t = F_s = 1$, $F_d = 0$ and $M = m$. The last term with F_d has no covariant counterpart and in fact will only appear (if at all) at two-loop order and beyond, as will be justified below. For the proper two-point function, the same arguments apply and we write

$$\Gamma_{\bar{q}q}(k) = \imath \left\{ k_0\gamma^0 A_t(k) - k_i\gamma^i A_s(k) - B_m(k) + k_0 k_i \gamma^0\gamma^i A_d(k) \right\} \tag{3.8}$$

and we will refer to A_t, A_s and B_m as the temporal, spatial and massive components, respectively. Again the last component (A_d) has no covariant counterpart and will only appear at two-loops or beyond. The relationship between the connected and proper two-point functions is supplied via the Legendre transform in standard fashion and we have

$$\Gamma_{\bar{q}q}(k) W_{\bar{q}q}(k) = 1. \tag{3.9}$$

In terms of the dressing functions, this gives

$$\begin{aligned} F_t &= \frac{(k_0^2 - \vec{k}^2 - m^2) A_t}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2 + k_0^2 \vec{k}^2 A_d^2}, \\ F_s &= \frac{(k_0^2 - \vec{k}^2 - m^2) A_s}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2 + k_0^2 \vec{k}^2 A_d^2}, \\ M &= \frac{(k_0^2 - \vec{k}^2 - m^2) B_m}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2 + k_0^2 \vec{k}^2 A_d^2}, \\ F_d &= \frac{(k_0^2 - \vec{k}^2 - m^2) A_d}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2 + k_0^2 \vec{k}^2 A_d^2}. \end{aligned} \tag{3.10}$$

Let us now discuss possible appearance of the genuinely noncovariant term corresponding to the dressing function A_d. In this work it will arise, if at all, from the one-loop perturbative form of the quark self-energy (see later for details).
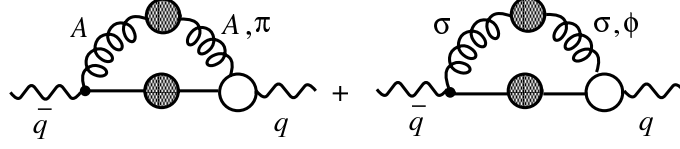
FIG. 1: Full nonperturbative diagram for the quark self-energy. Filled circles denote dressed propagators and empty circles denote dressed vertices. Springs denote connected (propagator) functions, solid lines denote quark propagators and wavy lines denote the external legs of the proper functions.

For now, we can anticipate that since the tree-level quark propagator does not contain a term with $k_0 k_i \gamma^0 \gamma^i$ and in the self-energy loop with two tree-level vertices we only have either two γ^0 or two γ^i matrices together (the gluon propagator is either purely temporal or spatial), then there is no one-loop contribution that has the overall structure $\gamma^0 \gamma^i$. This means that $A_d = 0$ at one-loop order perturbatively. For the rest of the dressing functions, we then get the simplified set of relations:

$$F_t = \frac{(k_0^2 - \vec{k}^2 - m^2)A_t}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2}, \quad F_s = \frac{(k_0^2 - \vec{k}^2 - m^2)A_s}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2}, \quad M = \frac{(k_0^2 - \vec{k}^2 - m^2)B_m}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2}. \tag{3.11}$$

We emphasize that these relations only hold up to one-loop perturbatively — in possible future studies it must be recognized that the fourth Dirac structure, $\gamma^0 \gamma^i$, may enter in a nontrivial fashion and that the set of relations Eq. (3.10) should be used.

Evaluation of the quark contributions to the W_{AA} and $W_{\sigma\sigma}$ propagators is a direct extension of the results already obtained in [14] for the Yang–Mills sector.

4. DERIVATION OF THE DYSON–SCHWINGER EQUATIONS

We start the derivation of the gap equation by taking the functional derivative of the quark field equation of motion (in configuration space), Eq. (2.14), with respect to $\imath q_w$:

$$<\imath\bar{q}_x \imath q_w> = \imath(\imath\gamma^0 \partial_{0x} + \imath\gamma^k \nabla_{kx} - m)\delta(x-w) - \int d^4 y d^4 z\, \delta(x-y)\delta(x-z)\left[\Gamma_{\bar{q}q\sigma}^{(0)a}\frac{\delta}{\delta \imath q_w}<\imath\rho_y^a \imath\bar{\chi}_z> + \Gamma_{\bar{q}qAj}^{(0)a}\frac{\delta}{\delta \imath q_w}<\imath J_{jy}^a \imath\bar{\chi}_z>\right]. \tag{4.1}$$

In the above we have used the configuration space definitions of the tree-level quark-gluon vertices (extracting the trivial δ-function dependence in configuration space) and omit those terms which will eventually vanish when the sources are set to zero. We use partial differentiation to calculate the terms in the bracket. Notice that simply because of the presence of the $\vec{\pi}$ and ϕ fields arising in the first order formalism, we must allow for the additional terms so generated (these terms will vanish when we consider the one-loop perturbative case):

$$\begin{aligned}
\frac{\delta}{\delta \imath q_w}<\imath\rho_y^a \imath\bar{\chi}_z> &= -\int d^4 v d^4 u <\imath\bar{\chi}_z \imath\chi_v><\imath\rho_y^a \imath\rho_u^b><\imath\bar{q}_v \imath q_w \imath\sigma_u^b> \\
&\quad -\int d^4 v d^4 u <\imath\bar{\chi}_z \imath\chi_v><\imath\rho_y^a \imath\kappa_u^b><\imath\bar{q}_v \imath q_w \imath\phi_u^b>,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\frac{\delta}{\delta \imath q_w}<\imath J_{jy}^a \imath\bar{\chi}_z> &= -\int d^4 v d^4 u <\imath\bar{\chi}_z \imath\chi_v><\imath J_{yj}^a \imath J_{uk}^b><\imath\bar{q}_v \imath q_w \imath A_{uk}^b> \\
&\quad -\int d^4 v d^4 u <\imath\bar{\chi}_z \imath\chi_v><\imath J_{yj}^a \imath K_{uk}^b><\imath\bar{q}_v \imath q_w \imath\pi_{uk}^b>.
\end{aligned} \tag{4.3}$$

Inserting the above expressions into Eq. (4.1) and Fourier transforming into momentum space we obtain the quark Dyson–Schwinger (or gap) equation

$$\begin{aligned}
\Gamma_{\bar{q}q}(k) &= \imath(\gamma^0 k_0 - \gamma^j k_j - m) + \int \bar{d}\omega\, \Gamma_{\bar{q}q\sigma}^{(0)a}(k, -\omega, \omega - k)W_{\bar{q}q}(\omega)\Gamma_{\bar{q}q\sigma}^b(\omega, -k, k - \omega)W_{\sigma\sigma}^{ab}(k - \omega) \\
&\quad + \int \bar{d}\omega\, \Gamma_{\bar{q}q\sigma}^{(0)a}(k, -\omega, \omega - k)W_{\bar{q}q}(\omega)\Gamma_{\bar{q}q\phi}^b(\omega, -k, k - \omega)W_{\sigma\phi}^{ab}(k - \omega)
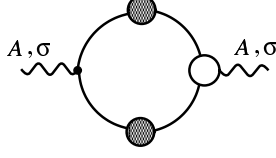\end{aligned}$$

FIG. 2: One-loop diagram for the quark contributions to the gluon proper two-point functions. Filled circles denote dressed propagators and empty circles denote dressed vertices. Solid lines denote quark propagators and wavy lines denote the external legs of the proper functions.

$$+ \int d\!\!\!/\,\omega \, \Gamma^{(0)a}_{\bar{q}qAi}(k, -\omega, \omega - k) W_{\bar{q}q}(\omega) \Gamma^b_{\bar{q}qAj}(\omega, -k, k - \omega) W^{ab}_{AAij}(k - \omega)$$

$$+ \int d\!\!\!/\,\omega \, \Gamma^{(0)a}_{\bar{q}qAi}(k, -\omega, \omega - k) W_{\bar{q}q}(\omega) \Gamma^b_{\bar{q}q\pi j}(\omega, -k, k - \omega) W^{ab}_{A\pi ij}(k - \omega), \tag{4.4}$$

where $d\!\!\!/\,\omega = d^4\omega/(2\pi)^4$. The self-energy corrections are presented diagrammatically in Fig. 1. We see that the π and ϕ fields do make a contribution thanks to the existence of mixed propagators in the first order formalism. But, as emphasized, these contributions will drop out at one-loop order perturbatively because of the absence of corresponding tree-level vertices, i.e., that there exist no direct interaction terms in the action between the quark fields and the auxiliary fields of the first order formalism.

We next consider the quark contributions to the proper two-point functions $\Gamma_{\sigma\sigma}$, $\Gamma_{\sigma A}$ and Γ_{AA}. Starting with the σ equation of motion Eq. (2.15) and following the same procedure as for the gap equation we derive the quark contribution to the proper two-point function $\Gamma_{\sigma\sigma}$ in configuration space (trace over Dirac and fundamental color indices):

$$<\imath\sigma^a_x \imath\sigma^b_w>_{(q)} = -\mathrm{Tr} \int d^4y d^4z d^4u d^4v \Gamma^{(0)a}_{\bar{q}q\sigma}(z, y, x) <\imath\bar{\chi}_y \imath\chi_u><\imath\bar{q}_u \imath q_v \imath\sigma^b_w><\imath\bar{\chi}_v \imath\chi_z> . \tag{4.5}$$

Taking the Fourier transform, we get in momentum space:

$$\Gamma^{ab}_{\sigma\sigma(q)}(k) = -\mathrm{Tr} \int d\!\!\!/\,\omega \, \Gamma^{(0)a}_{\bar{q}q\sigma}(\omega - k, -\omega, k) W_{\bar{q}q}(\omega) \Gamma^b_{\bar{q}q\sigma}(\omega, k - \omega, -k) W_{\bar{q}q}(\omega - k). \tag{4.6}$$

Similarly we obtain:

$$\Gamma^{ab}_{\sigma Ai(q)}(k) = -\mathrm{Tr} \int d\!\!\!/\,\omega \, \Gamma^{(0)a}_{\bar{q}q\sigma}(\omega - k, -\omega, k) W_{\bar{q}q}(\omega) \Gamma^b_{\bar{q}qAi}(\omega, k - \omega, -k) W_{\bar{q}q}(\omega - k), \tag{4.7}$$

$$\Gamma^{ab}_{AAij(q)}(k) = -\mathrm{Tr} \int d\!\!\!/\,\omega \, \Gamma^{(0)a}_{\bar{q}qAi}(\omega - k, -\omega, k) W_{\bar{q}q}(\omega) \Gamma^b_{\bar{q}qAj}(\omega, k - \omega, -k) W_{\bar{q}q}(\omega - k). \tag{4.8}$$

These loop contributions are shown collectively in Fig. 2.

5. ONE-LOOP PERTURBATIVE TWO-POINT FUNCTIONS

Let us now consider the one-loop perturbative expansions of the two-point functions derived in the previous section. Although so far the formalism has been presented in 4-dimensional Minkowski space, in order to evaluate the resulting loop integrals we have to make the analytic continuation to Euclidean space ($k_0 \to \imath k_4$), where we denote the temporal component of the Euclidean 4-momentum k_4 such that $k^2 = k_4^2 + \vec{k}^2$. Additionally, to regularize the integrals, dimensional regularization is employed with the Euclidean space integration measure

$$d\!\!\!/\,\omega = \frac{d\omega_4 d^d\vec{\omega}}{(2\pi)^{d+1}} \tag{5.1}$$

where $d = 3 - 2\varepsilon$ is the spatial dimension. To preserve the dimension of the action we must assign a dimension to the coupling through the replacement

$$g^2 \to g^2 \mu^\varepsilon, \tag{5.2}$$

where μ is the square of a non-vanishing mass scale (which may be later associated with a renormalization scale). The perturbative expansion of the two-point dressing function is generically written as:

$$\Gamma = \Gamma^{(0)} + g^2\Gamma^{(1)} \tag{5.3}$$

where the factor μ^ε is included in $\Gamma^{(1)}$ such that the new coupling and $\Gamma^{(1)}$ are dimensionless.

In the full nonperturbative gap equation, Eq. (4.4), we first insert the various tree-level vertices and propagators given by Eqs. (3.1), (3.2) and (3.5). Then we insert the general decomposition of the proper two-point functions, Eq. (3.8), occurring on the left-hand side of the gap equation, take the Dirac projection, solve the color and tensor algebra and lastly perform the Wick rotation. The one-loop temporal, spatial and massive components of the quark gap equation in Euclidean space read (recall that $C_F = (N_c^2 - 1)/2N_c$):

$$A_t(k) = 1 - g^2\mu^\varepsilon C_F \frac{1}{k_4^2} \int \dbar\omega \left\{ \frac{k_4\omega_4}{(\omega^2 + m^2)(\vec{k} - \vec{\omega})^2} - \frac{k_4\omega_4(d-1)}{(\omega^2 + m^2)(k-\omega)^2} \right\}, \tag{5.4}$$

$$A_s(k) = 1 - g^2\mu^\varepsilon C_F \frac{1}{\vec{k}^2} \int \dbar\omega \left\{ -\frac{2[\vec{k} \cdot (\vec{k} - \vec{\omega})][\vec{\omega} \cdot (\vec{k} - \vec{\omega})]}{(\omega^2 + m^2)(k-\omega)^2(\vec{k} - \vec{\omega})^2} - \frac{\vec{k} \cdot \vec{\omega}}{(\omega^2 + m^2)(\vec{k} - \vec{\omega})^2} + \frac{\vec{k} \cdot \vec{\omega}(3-d)}{(\omega^2 + m^2)(k-\omega)^2} \right\}, \tag{5.5}$$

$$B_m(k) = m + mg^2\mu^\varepsilon C_F \int \dbar\omega \left\{ \frac{1}{(\omega^2 + m^2)(\vec{k} - \vec{\omega})^2} + \frac{(d-1)}{(\omega^2 + m^2)(k-\omega)^2} \right\}. \tag{5.6}$$

As mentioned earlier, the possible contribution corresponding to the genuinely noncovariant dressing function A_d does not appear at one-loop. To evaluate the integrals occurring in Eq. (5.5), it is helpful to use the identity:

$$\vec{k} \cdot \vec{\omega} = \frac{1}{2} \left[k^2 + \omega^2 - (k-\omega)^2 \right] - k_4\omega_4, \tag{5.7}$$

which enables us to rewrite A_s as a combination of more straightforward integrals:

$$A_s(k) = 1 - g^2\mu^\varepsilon C_F \frac{1}{\vec{k}^2} \int \dbar\omega \left\{ -\frac{1}{2} \frac{(k^2 + m^2)^2}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} + \frac{2(k^2 + m^2)k_4\omega_4}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} + \frac{2\varepsilon\vec{k}^2 + 2k_4^2}{[(k-\omega)^2 + m^2]\omega^2} \right.$$
$$\left. + \frac{2\vec{k} \cdot \vec{\omega}(1-\varepsilon)}{[(k-\omega)^2 + m^2]\omega^2} - \frac{1}{2} \frac{m^2 + 3k^2}{[(k-\omega)^2 + m^2]\vec{\omega}^2} \right\}. \tag{5.8}$$

To determine the quark contributions to the various proper two-point gluon dressing functions given by Eqs. (4.6-4.8) (presented in Fig. 2), we again insert the tree-level factors given by Eqs. (3.1),(3.2) and (3.5), solve the color and tensor algebra and perform a Wick rotation. The one-loop integral expressions are:

$$\vec{k}^2\Gamma^{(1)}_{\sigma\sigma(q)}(k) = \mu^\varepsilon N_f 2 \int \dbar\omega \frac{\vec{\omega}^2 - \omega_4^2 - \vec{\omega} \cdot \vec{k} + \omega_4 k_4 + m^2}{(\omega^2 + m^2)[(k-\omega)^2 + m^2]}, \tag{5.9}$$

$$k_i k_4 \Gamma^{(1)}_{\sigma A(q)}(k) = \mu^\varepsilon N_f 4 \int \dbar\omega \frac{\omega_i\omega_4 - k_i\omega_4}{(\omega^2 + m^2)[(k-\omega)^2 + m^2]}, \tag{5.10}$$

$$\vec{k}^2 t_{ij}(\vec{k})\Gamma^{(1)}_{AA(q)}(k) + k_i k_j \bar{\Gamma}^{(1)}_{AA(q)}(k) = 2N_f\mu^\varepsilon \int \dbar\omega \frac{2\omega_i\omega_j - 2\omega_i k_j + \delta_{ij}(\omega_4 k_4 + \vec{\omega} \cdot \vec{k} - \omega_4^2 - \vec{\omega}^2 - m^2)}{(\omega^2 + m^2)[(k-\omega)^2 + m^2]}, \tag{5.11}$$

where $\Gamma^{(1)}_{AA(q)}$ and $\bar{\Gamma}^{(1)}_{AA(q)}$ are the transversal and longitudinal components of the proper two-point function $\Gamma^{(1)ab}_{AAij(q)}$ (given by Eq. (4.8)), respectively (see Ref. [14] for details of decomposition).

6. NONCOVARIANT MASSIVE LOOP INTEGRALS

In the one-loop expansions from the previous section, there are two types of integrals arising: those which can be solved using standard techniques (such as Schwinger parametrization, Mellin representation – for details, see Appendix A), and those which require a more esoteric approach. In this section we study the latter variety using a

technique based on differential equations and integration by parts developed previously [15]. We will consider the two integrals:

$$A_m(k_4^2, \vec{k}^2) = \int \frac{d\omega}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2},$$ (6.1)

$$A_m^4(k_4^2, \vec{k}^2) = \int \frac{d\omega \, \omega_4}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2}.$$ (6.2)

A. Derivation of the differential equations

Let us first write Eqs. (6.1) and (6.2) in the general form $(n = 0, 1)$

$$I^n(k_4^2, \vec{k}^2) = \int \frac{d\omega \, \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2}.$$ (6.3)

In this derivation k_4^2 and $\vec{k}^2$ are treated as variables whereas the mass, m, is treated as a parameter. The two first derivatives are:

$$k_4 \frac{\partial I^n}{\partial k_4} = \int \frac{d\omega \, \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} \left\{ -2\frac{k_4(k_4 - \omega_4)}{(k-\omega)^2 + m^2} \right\},$$ (6.4)

$$k_k \frac{\partial I^n}{\partial k_k} = \int \frac{d\omega \, \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} \left\{ -2\frac{\vec{k}\cdot(\vec{k} - \vec{\omega})}{(k-\omega)^2 + m^2} \right\}.$$ (6.5)

There are also two integration by parts identities:

$$0 = \int d\omega \frac{\partial}{\partial \omega_4} \frac{\omega_4^{n+1}}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} = \int \frac{d\omega \, \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} \left\{ n + 1 - 2\frac{\omega_4^2}{\omega^2} - 2\frac{\omega_4(\omega_4 - k_4)}{(k-\omega)^2 + m^2} \right\},$$ (6.6)

$$0 = \int d\omega \frac{\partial}{\partial \omega_i} \frac{\omega_i \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} = \int \frac{d\omega \, \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} \left\{ d - 2 - 2\frac{\vec{\omega}^2}{\omega^2} - 2\frac{\vec{\omega}\cdot(\vec{\omega} - \vec{k})}{(k-\omega)^2 + m^2} \right\}.$$ (6.7)

Adding these two expressions gives

$$0 = \int \frac{d\omega \, \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} \left\{ d + n - 3 - 2\frac{\omega \cdot (\omega - k)}{(k-\omega)^2 + m^2} \right\}.$$ (6.8)

Expanding the numerator factor, we can rewrite Eq. (6.8) as

$$0 = \int \frac{d\omega \, \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} \left\{ d + n - 4 + \frac{k^2 - \omega^2 + m^2}{(k-\omega)^2 + m^2} \right\}.$$ (6.9)

Combining this with Eq. (6.6) then yields:

$$\int \frac{d\omega \, \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} \left\{ -2\frac{k_4(k_4 - \omega_4)}{(k-\omega)^2 + m^2} \right\} = \int \frac{d\omega \, \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]\vec{\omega}^2} \left[\frac{2k_4^2}{k^2 + m^2}(n + d - 4) - n + 1 \right]$$

$$+ \frac{\vec{k}^2 + m^2}{k^2 + m^2} \int \frac{d\omega \, \omega_4^n}{[(k-\omega)^2 + m^2]^2\vec{\omega}^2} - 2\int \frac{d\omega \, \omega_4^n}{\omega^4[(k-\omega)^2 + m^2]} - 2\int \frac{d\omega \, \omega_4^n}{\omega^2[(k-\omega)^2 + m^2]^2}.$$ (6.10)

This leads to the temporal differential equations for A_m and A_m^4:

$$k_4 \frac{\partial A_m}{\partial k_4} = \left[1 + 2\frac{(d-4)k_4^2}{k^2 + m^2} \right] A_m + 2\frac{\vec{k}^2 + m^2}{k^2 + m^2} \int \frac{d\omega}{[(k-\omega)^2 + m^2]^2\vec{\omega}^2} - 2\int \frac{d\omega}{\omega^4[(k-\omega)^2 + m^2]}$$

$$- 2\int \frac{d\omega}{\omega^2[(k-\omega)^2 + m^2]^2},$$ (6.11)

$$k_4 \frac{\partial A_m^4}{\partial k_4} = 2\frac{(d-3)k_4^2}{k^2 + m^2} A_m^4 + 2\frac{\vec{k}^2 + m^2}{k^2 + m^2} \int \frac{d\omega \, \omega_4}{[(k-\omega)^2 + m^2]^2\vec{\omega}^2} - 2\int \frac{d\omega \, \omega_4}{\omega^4[(k-\omega)^2 + m^2]}$$

$$- 2\int \frac{d\omega \, \omega_4}{\omega^2[(k-\omega)^2 + m^2]^2}.$$ (6.12)

In the same manner, we derive the differential equations involving the spatial components:

$$
k_i \frac{\partial A_m}{\partial k_i} = \left[2 - d + 2 \frac{(d-4)\vec{k}^2}{k^2 + m^2} \right] A_m - \frac{2\vec{k}^2}{k^2 + m^2} \int \frac{đ\omega}{[(k-\omega)^2 + m^2]^2 \vec{\omega}^2} + 2 \int \frac{đ\omega}{\omega^4[(k-\omega)^2 + m^2]}
$$

$$
+ \ 2 \int \frac{đ\omega}{\omega^2[(k-\omega)^2 + m^2]^2}, \tag{6.13}
$$

$$
k_i \frac{\partial A_m^4}{\partial k_i} = \left[2 - d + 2 \frac{(d-3)\vec{k}^2}{k^2 + m^2} \right] A_m^4 - \frac{2\vec{k}^2}{k^2 + m^2} \int \frac{đ\omega \ \omega_4}{[(k-\omega)^2 + m^2]^2 \vec{\omega}^2} + 2 \int \frac{đ\omega \ \omega_4}{\omega^4[(k-\omega)^2 + m^2]}
$$

$$
+ \ 2 \int \frac{đ\omega \ \omega_4}{\omega^2[(k-\omega)^2 + m^2]^2}. \tag{6.14}
$$

It is in fact possible to write down a mass differential equation, but in the light of the method presented here this would not bring any new information. However, this third derivative is important because it provides an useful check of our solutions (for a detailed discussion, see Appendix B).

At this point, let us discuss how the differential equations for the massless integrals considered in Ref. [15] are regained. In the massless limit, there are potential ambiguities arising in the integrals appearing in Eqs. (6.11 -6.14), because in part, the limits $m \to 0$ and $\varepsilon \to 0$ do not interchange. Let us start by considering the following integral given by Eq. (A.16) (similar arguments apply to all the integrals appearing in the differential equations):

$$
I = \int \frac{đ\omega}{\vec{\omega}^2[(k-\omega)^2 + m^2]^2} = \frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(\frac{1}{2}-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(3/2-\varepsilon)} {}_2F_1\left(1, 1+\varepsilon; 3/2 - \varepsilon; -\frac{\vec{k}^2}{m^2}\right). \tag{6.15}
$$

It is useful here to invert the argument of the hypergeometric with the help of the formula (see, for instance, Ref. [19]):

$$
{}_2F_1(a,b;c;t) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-t)^{-a} {}_2F_1\left(a, 1-c+a; 1-b+a; \frac{1}{t}\right)
$$

$$
+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-t)^{-b} {}_2F_1\left(b, 1-c+b; 1-a+b; \frac{1}{t}\right). \tag{6.16}
$$

Then we have:

$$
I = \frac{1}{\vec{k}^2} \frac{[m^2]^{-\varepsilon}\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} {}_2F_1\left(1, \frac{1}{2}+\varepsilon; 1-\varepsilon; -\frac{m^2}{\vec{k}^2}\right) + \frac{[\vec{k}^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma\left(\frac{1}{2}-2\varepsilon\right)} {}_2F_1\left(1+\varepsilon, \frac{1}{2}+2\varepsilon; 1+\varepsilon; -\frac{m^2}{\vec{k}^2}\right). \tag{6.17}
$$

In the expression above, the problem of the non-interchangeable limits is seen explicitly in the first term. However, when all the integrals occurring in the various differential equations are put together, such terms explicitly cancel and only the second term of Eq. (6.17) (which leads to the correct massless limit) contributes.

Returning to the differential equations, we evaluate the standard integrals in terms of ε (see Appendix A) and with the notation $x = k_4^2$, $y = \vec{k}^2$ we obtain for A_m:

$$
2x \frac{\partial A_m}{\partial x} = \left[1 - 2(1+2\varepsilon)\frac{x}{x+y+m^2} \right] A_m + 2 \frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ \frac{y+m^2}{x+y+m^2} X {}_2F_1\left(1, 1+\varepsilon; 3/2-\varepsilon; -\frac{y}{m^2}\right) \right.
$$

$$
- \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)} {}_2F_1\left(2, 1+\varepsilon; 2-\varepsilon; -\frac{x+y}{m^2}\right) - Y {}_2F_1\left(1, 1+\varepsilon; 2-\varepsilon; -\frac{x+y}{m^2}\right) \right\}, \tag{6.18}
$$

$$
2y \frac{\partial A_m}{\partial y} = \left[-1 + 2\varepsilon - 2(1+2\varepsilon)\frac{y}{x+y+m^2} \right] A_m - 2 \frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ \frac{y}{x+y+m^2} X {}_2F_1\left(1, 1+\varepsilon; 3/2-\varepsilon; -\frac{y}{m^2}\right) \right.
$$

$$
- \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)} {}_2F_1\left(2, 1+\varepsilon; 2-\varepsilon; -\frac{x+y}{m^2}\right) - Y {}_2F_1\left(1, 1+\varepsilon; 2-\varepsilon; -\frac{x+y}{m^2}\right) \right\} \tag{6.19}
$$

and for the integral $A^4 = k_4 \overline{A}_m$:

$$
2x \frac{\partial \overline{A}_m}{\partial x} = \left[-1 - 4\varepsilon \frac{x}{x+y+m^2} \right] \overline{A}_m + 2 \frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ \frac{y+m^2}{x+y+m^2} X {}_2F_1\left(1, 1+\varepsilon; 3/2-\varepsilon; -\frac{y}{m^2}\right) \right.
$$

$$- \frac{Y}{2-\varepsilon} \left[{}_2F_1\left(2, 1+\varepsilon; 3-\varepsilon; -\frac{x+y}{m^2}\right) + (1-\varepsilon) {}_2F_1\left(1, 1+\varepsilon; 3-\varepsilon; -\frac{x+y}{m^2}\right) \right] \right\}, \tag{6.20}$$

$$2y\frac{\partial \overline{A}_m}{\partial y} = \left[-1 + 2\varepsilon - 4\varepsilon\frac{y}{x+y+m^2}\right] \overline{A}_m - 2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ \frac{y}{x+y+m^2}X_2F_1\left(1, 1+\varepsilon; 3/2-\varepsilon; -\frac{y}{m^2}\right) \right.$$
$$\left. - \frac{Y}{2-\varepsilon} \left[{}_2F_1\left(2, 1+\varepsilon; 3-\varepsilon; -\frac{x+y}{m^2}\right) + (1-\varepsilon) {}_2F_1\left(1, 1+\varepsilon; 3-\varepsilon; -\frac{x+y}{m^2}\right) \right] \right\}, \tag{6.21}$$

where

$$X = \frac{\Gamma(1/2-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(3/2-\varepsilon)}, \quad Y = \frac{\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)}. \tag{6.22}$$

B. Solving the differential equations

Let us first consider the integral A_m. By the same method as in [15], we make the following ansatz:

$$A_m(x,y) = F_{Am}(x,y)G_{Am}(x,y) \tag{6.23}$$

such that

$$2x\frac{\partial F_{Am}}{\partial x} = \left[1 - (2+4\varepsilon)\frac{x}{x+y+m^2}\right] F_{Am}, \tag{6.24}$$

$$2y\frac{\partial F_{Am}}{\partial y} = \left[-1 + 2\varepsilon - (2+4\varepsilon)\frac{y}{x+y+m^2}\right] F_{Am}, \tag{6.25}$$

$$F_{Am}2x\frac{\partial G_{Am}}{\partial x} = 2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ \frac{y+m^2}{x+y+m^2}X_2F_1\left(1, 1+\varepsilon; 3/2-\varepsilon; -\frac{y}{m^2}\right) \right.$$
$$\left. - \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)}{}_2F_1\left(2, 1+\varepsilon; 2-\varepsilon; -\frac{x+y}{m^2}\right) + Y_2F_1\left(1, 1+\varepsilon; 2-\varepsilon; -\frac{x+y}{m^2}\right) \right\}, \tag{6.26}$$

$$F_{Am}2y\frac{\partial G_{Am}}{\partial y} = 2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ -\frac{y}{x+y+m^2}X_2F_1\left(1, 1+\varepsilon; 3/2-\varepsilon; -\frac{y}{m^2}\right) \right.$$
$$\left. + \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)}{}_2F_1\left(2, 1+\varepsilon; 2-\varepsilon; -\frac{x+y}{m^2}\right) + Y_2F_1\left(1, 1+\varepsilon; 2-\varepsilon; -\frac{x+y}{m^2}\right) \right\}. \tag{6.27}$$

By inspection, it is simple to determine the solution for the two homogeneous equations, Eq. (6.24) and Eq. (6.25):

$$F_{Am}(x,y) = x^{1/2}y^{-1/2+\varepsilon}(x+y+m^2)^{-1-2\varepsilon}. \tag{6.28}$$

Since the mass m is treated as a parameter, the (dimensionful) solution, Eq. (6.28), may have an integration constant proportional to $[m^2]^{-1-\varepsilon}$. However, returning to the original equations, Eq. (6.24) and Eq. (6.25), we see that the only consistent solution is the one for which this constant vanishes.

For the function G_{Am} we make the following ansatz, which will be verified below ($z = x/y$):

$$G_{Am}(x,y) = G^0_{Am}(x,y) + \tilde{G}_{Am}(z). \tag{6.29}$$

The component $G^0_{Am}(x,y)$ can be found by adding the differential equations Eq. (6.26) and Eq. (6.27), which lead to:

$$x\frac{\partial G^0_{Am}}{\partial x} + y\frac{\partial G^0_{Am}}{\partial y} = \frac{1}{(4\pi)^{2-\varepsilon}} \left\{ \frac{m^2}{\sqrt{x(y+m^2)}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(\varepsilon) \right\}. \tag{6.30}$$

Because the function G^0_{Am} is multiplied by the function F_{Am} (which does not have an ε pole), the term of order $\mathcal{O}(\varepsilon)$ will not contribute. The solution of this equation is:

$$G^0_{Am}(x,y) = -\frac{2}{(4\pi)^{2-\varepsilon}} \left\{ \frac{1}{\sqrt{z}} \left[\sqrt{1+\frac{m^2}{y}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) - \ln y\right] + \mathcal{O}(\varepsilon) \right\} + \mathcal{C}_1. \tag{6.31}$$

Before we proceed to determine $\tilde{G}_{Am}$, we justify the ansatz for G_{Am}, given by Eq. (6.29). First we observe that:

$$2xF_{Am}\frac{\partial G_{Am}}{\partial x} = 2zF_{Am}\frac{\partial \tilde{G}_{Am}}{\partial z} + 2xF_{Am}\frac{\partial G^0_{Am}}{\partial x}. \tag{6.32}$$

Then we subtract the above equation, Eq. (6.32), from Eq. (6.26). This gives:

$$
\begin{aligned}
z\frac{\partial \tilde{G}_{Am}}{\partial z} &= \frac{1}{F_{Am}}\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{y+m^2}{x+y+m^2}X\,_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right) + Y\,_2F_1\left(1,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right)\right. \\
&\quad \left. - \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)}\,_2F_1\left(2,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right)\right\} - x\frac{\partial G^0_{Am}}{\partial x}.
\end{aligned}
\tag{6.33}
$$

Evaluation to the first order in ε is straightforward and we see that the right hand side of the above expression is only a function of the variable z. This allows us to write down a differential equation for $\tilde{G}_{Am}(z)$ in the form:

$$z\frac{\partial \tilde{G}_{Am}}{\partial z} = \frac{1}{(4\pi)^{2-\varepsilon}}\frac{1}{\sqrt{z}}\left\{\frac{1}{\varepsilon} - \gamma + \ln m^2 + \mathcal{O}(\varepsilon)\right\}, \tag{6.34}$$

from which we get immediately

$$\tilde{G}_{Am}(z) = -\frac{2}{(4\pi)^{2-\varepsilon}}\frac{1}{\sqrt{z}}\left\{\frac{1}{\varepsilon} - \gamma + \ln m^2 + \mathcal{O}(\varepsilon)\right\} + \mathcal{C}_2. \tag{6.35}$$

Returning to the original differential equations (6.24 - 6.27) with the function $G(x,y) = G^0_{Am}(x,y) + \tilde{G}_{Am}(z)$, we see that the only consistent solution is the one for which the overall constant $\mathcal{C}_1 + \mathcal{C}_2$ vanishes.

We may now put together the solutions Eqs. (6.28), (6.31) and (6.35) and write for the function A_m:

$$A_m(x,y) = \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{-\frac{2}{\varepsilon} + 2\gamma + 2\ln\left(\frac{x+y+m^2}{m^2}\right) - 2\sqrt{1+\frac{m^2}{y}}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(\varepsilon)\right\}. \tag{6.36}$$

We see that for $m^2 = 0$ we regain the result from [15] and that the singularities are located at $x + y + m^2 = 0$ (with $m^2, y \geq 0$). Two more useful checks arise from the study of the power expansion around $x = 0$ and the mass differential equation (for details, see Appendix B).

We now proceed in the same fashion to determine the function $\overline{A}_m(x,y) = F_{\overline{A}m}(x,y)G_{\overline{A}m}(x,y)$. The resulting partial differential equations are in this case:

$$2x\frac{\partial F_{\overline{A}m}}{\partial x} = \left[-1 - 4\varepsilon\frac{x}{x+y+m^2}\right]F_{\overline{A}m}, \tag{6.37}$$

$$2y\frac{\partial F_{\overline{A}m}}{\partial y} = \left[-1 + 2\varepsilon - 4\varepsilon\frac{y}{x+y+m^2}\right]F_{\overline{A}m}, \tag{6.38}$$

$$
\begin{aligned}
F_{\overline{A}m}2x\frac{\partial G_{\overline{A}m}}{\partial x} &= 2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{y+m^2}{x+y+m^2}X\,_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right)\right. \\
&\quad \left. - \frac{Y}{2-\varepsilon}\left[\,_2F_1\left(2,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right) + (1-\varepsilon)\,_2F_1\left(1,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right)\right]\right\},
\end{aligned}
\tag{6.39}
$$

$$
\begin{aligned}
F_{\overline{A}m}2y\frac{\partial G_{\overline{A}m}}{\partial y} &= 2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{-\frac{y}{x+y+m^2}X\,_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right)\right. \\
&\quad \left. + \frac{Y}{2-\varepsilon}\left[\,_2F_1\left(2,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right) + (1-\varepsilon)\,_2F_1\left(1,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right)\right]\right\},
\end{aligned}
\tag{6.40}
$$

with X, Y defined previously. The solution to the first pair is

$$F_{\overline{A}m}(x,y) = x^{-1/2}y^{-1/2+\varepsilon}(x+y+m^2)^{-2\varepsilon}. \tag{6.41}$$

For brevity, in the above expression and also in the derivation of the function $G_{\overline{A}m} = G^0_{\overline{A}m} + \tilde{G}_{\overline{A}m}$ (the analogue of G_{Am}) we omit the constants of integration – they vanish as in the case of the functions F_{Am} and G_{Am}.

As before, for $G_{\overline{A}m}(x,y)$ we make the ansatz:

$$G_{\overline{A}m}(x,y) = G^0_{\overline{A}m}(x,y) + \tilde{G}_{\overline{A}m}(z).\tag{6.42}$$

In the limit $\varepsilon \to 0$, the component $G^0_{\overline{A}m}(x,y)$ is determined from the differential equation:

$$x\frac{\partial G^0_{\overline{A}m}}{\partial x} + y\frac{\partial G^0_{\overline{A}m}}{\partial y} = \frac{1}{(4\pi)^{2-\varepsilon}}\left\{\frac{\sqrt{x}}{x+y+m^2}\frac{m^2}{\sqrt{y+m^2}}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(\varepsilon)\right\}.\tag{6.43}$$

The solution of this equation is:

$$G^0_{\overline{A}m}(x,y) = \frac{1}{(4\pi)^{2-\varepsilon}}\left\{\imath\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}-\imath\sqrt{z}}{\sqrt{1+\frac{m^2}{y}}+\imath\sqrt{z}}\right)\ln\left(\frac{\imath\sqrt{z}+1}{\imath\sqrt{z}-1}\right)\right.$$

$$\left.-\imath\mathrm{Li}_2\left(\frac{1-\imath\sqrt{z}}{1+\imath\sqrt{z}}\cdot\frac{\sqrt{1+\frac{m^2}{y}}-\imath\sqrt{z}}{\sqrt{1+\frac{m^2}{y}}+\imath\sqrt{z}}\right) + \imath\mathrm{Li}_2\left(\frac{1+\imath\sqrt{z}}{1-\imath\sqrt{z}}\cdot\frac{\sqrt{1+\frac{m^2}{y}}-\imath\sqrt{z}}{\sqrt{1+\frac{m^2}{y}}+\imath\sqrt{z}}\right) + \mathcal{O}(\varepsilon)\right\},\tag{6.44}$$

where $\mathrm{Li}_2(z)$ is the dilogarithmic function [20]:

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}dt.\tag{6.45}$$

As before, we check that the ansatz for $G_{\overline{A}m}(x,y)$ given in Eq. (6.42) is correct and derive the differential equation for the function $\tilde{G}_{\overline{A}m}$, in the limit $\varepsilon \to 0$:

$$z\frac{\partial \tilde{G}_{\overline{A}m}}{\partial z} = \frac{1}{(4\pi)^{2-\varepsilon}}\left\{\frac{\sqrt{z}}{z+1}\left[\ln(1+z) - \ln z - 2\ln 2\right] + \mathcal{O}(\varepsilon)\right\}.\tag{6.46}$$

The result we leave for the moment in the form:

$$\tilde{G}_{\overline{A}m}(z) = \frac{1}{(4\pi)^{2-\varepsilon}}\left\{-4\ln 2\arctan(\sqrt{z}) + \int_0^z \frac{dt}{\sqrt{t}(1+t)}\ln(1+t) - \int_0^z \frac{dt}{\sqrt{t}(1+t)}\ln t + \mathcal{O}(\varepsilon)\right\}.\tag{6.47}$$

With the solutions, Eqs. (6.41), (6.44) and (6.47), after some further manipulation we can write down the following simplified expression for the integral A_m^4:

$$A_m^4(x,y) = k_4\frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{(1+z+\frac{m^2}{y})}{\sqrt{z}}\left\{2\ln\left(\frac{m^2}{y}\right)\arctan(\sqrt{z}) + 2\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right)\arctan\left(\frac{\sqrt{z}}{\sqrt{\frac{m^2}{y}+1}}\right)\right.$$

$$\left.-\int_0^z \frac{dt}{\sqrt{t}(1+t)}\ln\left(1+t+\frac{m^2}{y}\right) + \mathcal{O}(\varepsilon)\right\},\tag{6.48}$$

with the integral

$$\int_0^z \frac{dt}{\sqrt{t}(1+t)}\ln\left(1+t+\frac{m^2}{y}\right) = \pi\ln 2 - \imath\ln\left(\frac{1-\imath\sqrt{z}}{1+\imath\sqrt{z}}\right)2\ln 2$$

$$+ \imath\ln\left(\frac{1-\imath\sqrt{z}}{1+\imath\sqrt{z}}\frac{\sqrt{1+\frac{m^2}{y}}-\imath\sqrt{z}}{\sqrt{1+\frac{m^2}{y}}+\imath\sqrt{z}}\right)\ln\left(\sqrt{1+\frac{m^2}{y}}+1\right) + \imath\ln\left(\frac{1-\imath\sqrt{z}}{1+\imath\sqrt{z}}\frac{\sqrt{1+\frac{m^2}{y}}+\imath\sqrt{z}}{\sqrt{1+\frac{m^2}{y}}-\imath\sqrt{z}}\right)\ln\left(\sqrt{1+\frac{m^2}{y}}-1\right)$$

$$- \imath\ln(\sqrt{z}-\imath)\left[\ln 2 + \ln(1+z+\frac{m^2}{y}) - \ln(1-\imath\sqrt{z}) - \frac{1}{2}\ln(\sqrt{z}-\imath)\right] - \imath\mathrm{Li}_2\left(\frac{1}{2}-\frac{\imath}{2}\sqrt{z}\right) + \imath\mathrm{Li}_2\left(\frac{1}{2}+\frac{\imath}{2}\sqrt{z}\right)$$

$$+ \imath\ln(\sqrt{z}+\imath)\left[\ln 2 + \ln(1+z+\frac{m^2}{y}) - \ln(1+\imath\sqrt{z}) - \frac{1}{2}\ln(\sqrt{z}+\imath)\right] - \imath\mathrm{Li}_2\left(\frac{\imath+\sqrt{z}}{-\imath+\sqrt{z}}\right) + \imath\mathrm{Li}_2\left(\frac{-\imath+\sqrt{z}}{\imath+\sqrt{z}}\right)$$

$$+ \imath\mathrm{Li}_2\left(\frac{\imath\sqrt{1+\frac{m^2}{y}}+\sqrt{z}}{-\imath+\sqrt{z}}\right) - \imath\mathrm{Li}_2\left(\frac{-\imath\sqrt{1+\frac{m^2}{y}}+\sqrt{z}}{\imath+\sqrt{z}}\right) + \imath\mathrm{Li}_2\left(\frac{-\imath\sqrt{1+\frac{m^2}{y}}+\sqrt{z}}{-\imath+\sqrt{z}}\right) + \imath\mathrm{Li}_2\left(\frac{\imath\sqrt{1+\frac{m^2}{y}}+\sqrt{z}}{\imath+\sqrt{z}}\right).$$

$$\tag{6.49}$$

We see that for $m^2 = 0$ we get the correct limit for the function A_m^4. We also mention that the singularities are located at $x + y + m^2 = 0$ and the apparent singularities at $z = -1$ (i.e., $x + y = 0$) in the expression Eq. (6.48) are canceling out. This can be easily seen by making a series expansion of Eq. (6.48) around $z = -1$:

$$A_m^4 \overset{z \to -1}{=} \frac{1}{\sqrt{z}} \left[\frac{y}{m^2} - \frac{z+1}{2} \left(\frac{y}{m^2} \right)^2 + \mathcal{O}\left((z+1)^3 \right) \right] - \frac{\sqrt{1 + \frac{m^2}{y}}}{\sqrt{z}(z + 1 + \frac{m^2}{y})} \ln \left(\frac{\sqrt{1 + \frac{m^2}{y}} + 1}{\sqrt{1 + \frac{m^2}{y}} - 1} \right) + \mathcal{O}(\varepsilon). \tag{6.50}$$

Again, the result Eq. (6.48) has been checked by performing an expansion around $x = 0$ and by studying the mass differential equation (see Appendix B).

7. PERTURBATIVE RESULTS IN THE LIMIT $\varepsilon \to 0$

We can now collect together the results and write down the one-loop perturbative expressions for the two-point functions. In Eqs. (5.4), (5.6) and (5.8) we insert the corresponding integrals (derived in the previous section and Appendix A) and find for the temporal, spatial and massive components of the quark gap equation, in the limit $\varepsilon \to 0$:

$$A_t(k) = 1$$
$$+ \frac{C_F g^2}{(4\pi)^{2-\varepsilon}} \left\{ \frac{1}{\varepsilon} - \gamma - \ln \frac{m^2}{\mu} + 1 - \frac{m^2}{k^2} + \left(\frac{m^4}{k^4} - 1 \right) \ln \left(1 + \frac{k^2}{m^2} \right) + \mathcal{O}(\varepsilon) \right\}, \tag{7.1}$$

$$A_s(k) = 1$$
$$+ \frac{C_F g^2}{(4\pi)^{2-\varepsilon}} \left\{ \frac{1}{\varepsilon} - \gamma - \ln \frac{m^2}{\mu} + 1 + 8 \frac{k^2}{\vec{k}^2} + 4 \frac{m^2}{\vec{k}^2} - \frac{m^2}{k^2} + \left(1 + \frac{m^2}{k^2} \right) \left(4 \frac{k^2}{\vec{k}^2} - 1 + \frac{m^2}{k^2} \right) \ln \left(1 + \frac{k^2}{m^2} \right) \right.$$
$$\left. - \left(4 \frac{k^2}{\vec{k}^2} + 2 \frac{m^2}{\vec{k}^2} \right) \sqrt{1 + \frac{m^2}{\vec{k}^2}} \ln \left(\frac{\sqrt{1 + \frac{m^2}{\vec{k}^2}} + 1}{\sqrt{1 + \frac{m^2}{\vec{k}^2}} - 1} \right) - 2 \frac{k_4^2}{\vec{k}^4} (k^2 + m^2) f_m \left(k_4^2, \vec{k}^2 \right) + \mathcal{O}(\varepsilon) \right\}, \tag{7.2}$$

$$B_m(k) = m$$
$$+ m \frac{C_F g^2}{(4\pi)^{2-\varepsilon}} \left\{ \frac{4}{\varepsilon} - 4\gamma - 4 \ln \frac{m^2}{\mu} + 10 - 2 \sqrt{1 + \frac{m^2}{\vec{k}^2}} \ln \left(\frac{\sqrt{1 + \frac{m^2}{\vec{k}^2}} + 1}{\sqrt{1 + \frac{m^2}{\vec{k}^2}} - 1} \right) - 2 \left(1 + \frac{m^2}{k^2} \right) \ln \left(1 + \frac{k^2}{m^2} \right) + \mathcal{O}(\varepsilon) \right\}, \tag{7.3}$$

where the function $f_m(x, y)$ is given by ($z = x/y \equiv k_4^2/\vec{k}^2$):

$$f_m(x, y) = \frac{2}{\sqrt{z}} \ln \left(\frac{m^2}{y} \right) \arctan \left(\sqrt{z} \right) + \frac{2}{\sqrt{z}} \ln \left(\frac{\sqrt{1 + \frac{m^2}{y}} + 1}{\sqrt{1 + \frac{m^2}{y}} - 1} \right) \arctan \left(\frac{\sqrt{z}}{\sqrt{\frac{m^2}{y} + 1}} \right)$$
$$- \int_0^1 \frac{dt}{\sqrt{t}(1 + zt)} \ln \left(1 + zt + \frac{m^2}{y} \right). \tag{7.4}$$

The last integral has been rewritten using the identity:

$$\frac{1}{\sqrt{z}} \int_0^z \frac{dt}{\sqrt{t}(1 + t)} \ln \left(1 + t + \frac{m^2}{y} \right) = \int_0^1 \frac{dt}{\sqrt{t}(1 + zt)} \ln \left(1 + zt + \frac{m^2}{y} \right). \tag{7.5}$$

As a useful check, we can set $m = 0$ and show that the results for the temporal and spatial components are in agreement with the calculation performed independently using the one-loop massless integrals derived in Ref. [15]. As has been shown in the previous section, in the noncovariant integrals the singularities appear at $x + y + m^2 = 0$. It is easy to see that the standard integrals also have the same singularity structure. Because of the absence of singularities in both the Euclidean and spacelike Minkowski regions, we can see that the validity of the Wick rotation is justified.

Having calculated the dressing functions for the quark proper two-point Green's function, we are now able to discuss the structure of the propagator. In Eq. (3.11) we first analyze the denominator factor. Let us denote (in Euclidean space):

$$D(k) = k_4^2 A_t^2(k) + \vec{k}^2 A_s^2(k) + B_m^2(k). \tag{7.6}$$

Inserting the expressions from Eqs. (7.1), (7.2) and (7.3) into the above equation, we have:

$$D(k) = k^2 + m^2 \left\{ 1 + 6 \frac{g^2 C_F}{(4\pi)^{2-\varepsilon}} \left[\frac{1}{\varepsilon} - \gamma - \ln \frac{m^2}{\mu} + \frac{4}{3} \right] \right\} + (k^2 + m^2) \frac{2 C_F g^2}{(4\pi)^{2-\varepsilon}} \left\{ \frac{1}{\varepsilon} - \gamma - \ln \frac{m^2}{\mu} \right\}$$

$$+ (k^2 + m^2) \frac{2 C_F g^2}{(4\pi)^{2-\varepsilon}} \left\{ 9 + \left(3 - \frac{m^2}{k^2} \right) \ln \left(1 + \frac{k^2}{m^2} \right) - 4 \sqrt{1 + \frac{m^2}{\vec{k}^2}} \ln \left(\frac{\sqrt{1 + \frac{m^2}{\vec{k}^2}} + 1}{\sqrt{1 + \frac{m^2}{\vec{k}^2}} - 1} \right) - 2 \frac{k_4^2}{\vec{k}^2} f_m(k_4^2, \vec{k}^2) \right\}. \quad (7.7)$$

We define the renormalized mass, m_R, via:

$$m^2 = Z_m^2 m_R^2 \quad \text{with} \quad Z_m^2 = 1 - 6 \frac{g^2 C_F}{(4\pi)^{2-\varepsilon}} \left\{ \frac{1}{\varepsilon} - \gamma - \ln \frac{m^2}{\mu} + \frac{4}{3} \right\}. \quad (7.8)$$

The expression for $D(k)$, Eq. (7.7), then contains explicitly the overall factor $k^2 + m_R^2$, meaning that the simple pole mass of the quark emerges, just as it does in covariant gauges. The singularity structure of the remaining part is such that there are no non-analytic structures for spacelike or Euclidean momenta. Moreover, we see that the renormalization factor, Z_m, which should be a gauge invariant quantity (it defines the physical perturbative pole mass) agrees with the result obtained in covariant gauges [21].

Because of the Dirac structure, it is more convenient to write the quark propagator in Minkowski space. We have shown that the analytic continuation of the functions A_t, A_s, B_m back into the Minkowski space is allowed and this enables us simply to write (note that also in $D(k)$ we must also analytically continue $k_4^2 \to -k_0^2$):

$$W_{\bar{q}q}(k) = \imath \left\{ \gamma^0 k_0 A_t(k) - \gamma^i k_i A_s(k) + B_m(k) \right\} D^{-1}(k). \quad (7.9)$$

Inserting the denominator factor, Eq. (7.7), in the limit $\varepsilon \to 0$ and replacing the mass with its renormalized counterpart, the above expression gives:

$$W_{\bar{q}q}(k) = -\frac{i}{k_0^2 - \vec{k}^2 - m_R^2} \left\{ (\not{k} + m_R) \left[1 - C_F \frac{g^2}{(4\pi)^{2-\varepsilon}} \left(\frac{1}{\varepsilon} - \gamma \right) \right] + \text{ finite terms} \right\}. \quad (7.10)$$

We can thus write down for the quark propagator:

$$W_{\bar{q}q}(k) = -\frac{\imath (\not{k} + m_R)}{k_0^2 - \vec{k}^2 - m_R^2} Z_2 + \text{ finite terms} \quad (7.11)$$

and identify the renormalization constant (omitting the prescription dependent constants)

$$Z_2 = 1 - \frac{g^2 C_F}{(4\pi)^{2-\varepsilon}} \left(\frac{1}{\varepsilon} - \gamma \right). \quad (7.12)$$

Turning to the quark loop contributions to the gluon two-point proper functions, in evaluating the integral structure of Eqs. (5.9), (5.10) and (5.11) we observe the following relations (in Euclidean space):

$$\Gamma_{\sigma\sigma(q)}^{(1)}(k) = \Gamma_{\sigma A(q)}^{(1)}(k) = -\frac{\vec{k}^2}{k^2} \Gamma_{AA(q)}^{(1)}(k) = -\frac{\vec{k}^2}{k_4^2} \bar{\Gamma}_{AA,q}^{(1)}(k) = I(k_4^2, \vec{k}^2), \quad (7.13)$$

where the integral $I(k_4^2, \vec{k}^2)$ reads (using the results of Appendix A), as $\varepsilon \to 0$:

$$I(k_4^2, \vec{k}^2) = \frac{N_f}{(4\pi)^{2-\varepsilon}} \left\{ -\frac{2}{3} \left[\frac{1}{\varepsilon} - \gamma - \ln \frac{k^2}{\mu} \right] - \frac{10}{9} + \frac{2}{3} \sqrt{1 + \frac{4m^2}{k^2}} \left(1 - 2 \frac{m^2}{k^2} \right) \ln \left(\frac{\sqrt{1 + \frac{4m^2}{k^2}} + 1}{\sqrt{1 + \frac{4m^2}{k^2}} - 1} \right) \right.$$

$$\left. + \frac{2}{3} \left(4 \frac{m^2}{k^2} + \ln \frac{m^2}{k^2} \right) + \mathcal{O}(\varepsilon) \right\}. \quad (7.14)$$

The above integral agrees with the results obtained in covariant gauges (see for instance [21]). This is hardly surprising, since at one-loop level the quark loop as a whole is unchanged from its covariant counterpart — what is different is that the various degrees of freedom (temporal and spatial) are being separated into the corresponding proper two-point functions, i.e., Γ_{AA}, $\Gamma_{A\sigma}$ and $\Gamma_{\sigma\sigma}$.

The one-loop gluon propagator dressing functions we construct by writing $D = D^{(0)} + g^2 D^{(1)}$. As mentioned previously, in the first order formalism we have to account for the presence of the additional $\vec{\pi}, \phi$ and ghost fields and the corresponding propagators (for example $D_{A\pi}$). Whilst the quarks only contribute to three of the gluon proper two-point functions ($\Gamma_{AA}, \Gamma_{A\sigma}$ and $\Gamma_{\sigma\sigma}$) at one-loop, there will be contributions to many more of the various connected (propagator) two-point functions. The relationship between the connected and proper gluon two-point functions in the first order formalism is detailed in Ref. [15]. The full set of quark contributions to these gluonic type propagators is:

$$D_{AA(q)}^{(1)}(k) = D_{\sigma\sigma(q)}^{(1)}(k) = \Gamma_{\sigma\sigma(q)}^{(1)}(k) = I(k_4^2, \vec{k}^2),$$

$$D_{A\pi(q)}^{(1)}(k) = -\frac{\vec{k}^2}{k_4^2} D_{\pi\pi(q)}^{(1)}(k) = D_{\sigma\phi(q)}^{(1)}(k) = -D_{\phi\phi(q)}^{(1)}(k) = I(k_4^2, \vec{k}^2). \tag{7.15}$$

At this point we are able to identify the first coefficient of the perturbative β-function. As is well known in Landau gauge, a renormalization group invariant running coupling can be defined through the following perturbative combination of gluon and ghost propagator dressing functions [22]:

$$g^2 D_{AA} D_c^2 \sim g^2 \left[1 + \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \left(\frac{11 N_c}{3} - \frac{2 N_f}{3} \right) \right]. \tag{7.16}$$

At one-loop in perturbation theory, the coefficient of the $1/\varepsilon$ pole above is simply minus the first coefficient of the β-function ($\beta_0 = -11 N_c/3 + 2 N_f/3$). By inspecting the relations Eq. (7.15) and those obtained in Ref. [15] for the propagators D_{AA} and D_c, we see that the same result is achieved in Coulomb gauge. In Coulomb gauge, a second renormalization group invariant combination of propagators appears and is given by $g^2 D_{\sigma\sigma}$ [4]. Again, combining our results Eq. (7.15) and those obtained in [15], we see that indeed the coefficient of $1/\varepsilon$ agrees with this.

8. SUMMARY AND OUTLOOK

In this paper, the quark contributions to the Dyson–Schwinger equations of QCD have been derived within the Coulomb gauge first order formalism and perturbative results have been presented. The set of Feynman rules has been derived and the general form of the two-point functions have been established. The quark gap equation and the quark loop contributions to the Dyson–Schwinger equations concerning the gluon proper two-point functions have been explicitly derived. A one-loop perturbative calculation has been performed, for the quark gap equation, as well as for the quark contributions to the gluon proper two-point functions. The required noncovariant massive integrals have been obtained, using techniques based on differential equations and integration by parts. The various two-point dressing functions and propagators have been evaluated in the limit $\varepsilon \to 0$. The validity of the analytic continuation between Minkowski and Euclidean space has been verified. The quark mass and propagator have been renormalized and it is seen that the one-loop result for the gauge invariant quark mass renormalization coefficient agrees explicitly with the result obtained in linear covariant gauges. The correct one-loop coefficient for the β function has also been obtained.

The natural continuation of this work is the perturbative evaluation of vertex functions of the theory. The Mellin-Barnes parametrization or perhaps a generalization of the differential equation method to the three-point integrals would be possible ways to proceed. Also, the construction of scattering matrix elements would be another interesting topic.

APPENDIX A: STANDARD MASSIVE INTEGRALS

Consider the integral:

$$J_m(k^2) = \int \frac{đ\omega}{[\omega^2 + m^2]^\mu [(k-\omega)^2 + m^2]^\nu}. \tag{A.1}$$

In the case $\mu = \nu = 1$ this gives the scalar integral associated with, for example, the fermion loop in quantum electrodynamics [21]. We present here a method to evaluate such integrals for arbitrary denominator powers (developed originally in Ref. [23]) and generalize to the various additional noncovariant integrals.

We start by writing the Taylor expansion of the massive propagator in terms of a hypergeometric function in the following way:

$$\frac{1}{[\omega^2 + m^2]^\mu} = \frac{1}{[\omega^2]^\mu} {}_1F_0\left(\mu; -\frac{m^2}{\omega^2}\right). \tag{A.2}$$

Now, the idea is to use the Mellin-Barnes representation of the hypergeometric function ${}_1F_0(\mu; z)$:

$$ {}_1F_0(\mu; z) = \frac{1}{\Gamma(\mu)} \frac{1}{2\pi\imath} \int\limits_{-\imath\infty}^{\imath\infty} ds(-z)^s \Gamma(-s)\Gamma(\mu + s), \tag{A.3}$$

where the contour in the complex plane separates the left poles of the Γ functions from the right poles. A first advantage of this representation is that the "mass term" gets separated from the massless propagator and the remaining integrals can be calculated with the Cauchy residue theorem, as we shall see below. Also, we note that the results can be written as a function of either k^2/m^2, or m^2/k^2 (expansions thereof are of interest in studying various momentum regimes). This we do by using the well-known formulas of analytic continuation of the hypergeometric function (for an extended discussion, see [23]).

Applying Eq. (A.3) to the massive propagator we can rewrite the integral $J_m(k^2)$ as:

$$J_m(k^2) = \frac{1}{(2\pi\imath)^2} \frac{1}{\Gamma(\mu)\Gamma(\nu)} \iint\limits_{-\imath\infty}^{\imath\infty} dsdt(m^2)^{s+t} \Gamma(-s)\Gamma(-t)\Gamma(\mu + s)\Gamma(\nu + t) \int \frac{d\omega}{(\omega^2)^{\mu+s}[(k-\omega)^2]^{\nu+t}}. \tag{A.4}$$

Inserting the general result for the massless integral (an explicit derivation can be found in Ref. [15]):

$$\int \frac{d\omega}{[\omega^2]^\mu[(k-\omega)^2]^\nu} = \frac{[k^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\mu)\Gamma(\nu)} \frac{\Gamma(2-\mu-\varepsilon)\Gamma(2-\nu-\varepsilon)}{\Gamma(4-\mu-\nu-2\varepsilon)}, \tag{A.5}$$

we get for the integral J_m:

$$J_m(k^2) = \frac{[k^2]^{2-\nu-\mu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{1}{(2\pi\imath)^2} \frac{1}{\Gamma(\mu)\Gamma(\nu)} \iint\limits_{-\imath\infty}^{\imath\infty} dsdt \left(\frac{m^2}{k^2}\right)^{s+t} \Gamma(-s)\Gamma(-t)\Gamma(2-\varepsilon-\mu-s)\Gamma(2-\varepsilon-\nu-t)$$

$$\times \frac{\Gamma(\mu+\nu+s+t-2+\varepsilon)}{\Gamma(4-2\varepsilon-\mu-\nu-s-t)}. \tag{A.6}$$

With the change of variable $t = 2 - \varepsilon - \mu - \nu - u - s$ (for such a replacement, the left and right poles of the Γ function are simply interchanged and therefore the condition of separating the poles is not contradicted) we obtain:

$$J_m(k^2) = \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{1}{(2\pi\imath)} \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int\limits_{-\imath\infty}^{\imath\infty} du \left(\frac{m^2}{k^2}\right)^{-u} \frac{\Gamma(-u)}{\Gamma(2-\varepsilon+u)}$$

$$\times \frac{1}{(2\pi\imath)} \int\limits_{-\imath\infty}^{\imath\infty} ds\Gamma(-s)\Gamma(2-\varepsilon-\mu-s)\Gamma(-2+\varepsilon+\nu+\mu+u+s)\Gamma(\mu+u+s). \tag{A.7}$$

To evaluate the integral over s we use the Barnes Lemma:

$$\frac{1}{(2\pi\imath)} \int\limits_{-\imath\infty}^{\imath\infty} ds\ \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \tag{A.8}$$

and for the integral Eq. (A.7) it follows immediately that

$$J_m(k^2) = \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{1}{(2\pi\imath)} \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int\limits_{-\imath\infty}^{\imath\infty} du \left(\frac{m^2}{k^2}\right)^{-u} \frac{\Gamma(-u)\Gamma(\mu+u)\Gamma(\nu+u)\Gamma(\mu+\nu-2+\varepsilon+u)}{\Gamma(\mu+\nu+2u)}. \tag{A.9}$$

Closing the integration contour on the right we have:

$$J_m(k^2) = \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{1}{(2\pi\imath)} \frac{1}{\Gamma(\mu)\Gamma(\nu)} (2\pi\imath) \sum_{j=0}^{\infty} \left(-\frac{m^2}{k^2}\right)^{-j} \frac{1}{j!} \frac{\Gamma(\mu+j)\Gamma(\nu+j)\Gamma(\mu+\nu-2+\varepsilon+j)}{\Gamma(\mu+\nu+2j)}. \quad (A.10)$$

With the help of the duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \quad (A.11)$$

we can rewrite J_m as:

$$J_m(k^2) = \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(\mu+\nu-2+\varepsilon)}{\Gamma(\mu+\nu)}$$
$$\sum_{j=0}^{\infty} \left(-\frac{m^2}{k^2}\right)^{-j} \frac{1}{2^{2j}} \frac{1}{j!} \frac{\Gamma(\mu+j)}{\Gamma(\mu)} \frac{\Gamma(\nu+j)}{\Gamma(\nu)} \frac{\Gamma(\mu+\nu-2+\varepsilon+j)}{\Gamma(\mu+\nu-2+\varepsilon)} \frac{\Gamma(\frac{\mu+\nu}{2})}{\Gamma(\frac{\mu+\nu}{2}+j)} \frac{\Gamma(\frac{\mu+\nu+1}{2})}{\Gamma(\frac{\mu+\nu+1}{2}+j)}. \quad (A.12)$$

The sum is clearly a representation of the hypergeometric ${}_3F_2(a,b,c;d,e;z)$ (see [19]) and we finally obtain:

$$J_m(k^2) = \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(\mu+\nu-2+\varepsilon)}{\Gamma(\mu+\nu)} {}_3F_2\left(\mu,\nu,\mu+\nu-2+\varepsilon; \frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; -\frac{k^2}{4m^2}\right). \quad (A.13)$$

A trivial computation shows that the result Eq. (A.13) is consistent with the known results in the limit $m=0$. All we have to do is to invert the argument of the hypergeometric according to the formula (see, for example, Ref. [24])

$${}_3F_2(a_1,a_2,a_3;b_1,b_2;z) = \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \left\{ \frac{\Gamma(a_1)\Gamma(a_2-a_1)\Gamma(a_3-a_1)}{\Gamma(b_1-a_1)\Gamma(b_2-a_1)}(-z)^{-a_1} \right.$$
$$\times {}_3F_2\left(a_1, a_1-b_1+1, a_1-b_2+1; a_1-a_2+1, a_1-a_3+1; \frac{1}{z}\right)$$
$$+ \frac{\Gamma(a_2)\Gamma(a_1-a_2)\Gamma(a_3-a_2)}{\Gamma(b_1-a_2)\Gamma(b_2-a_2)}(-z)^{-a_2} {}_3F_2\left(a_2, a_2-b_1+1, a_2-b_2+1; -a_1+a_2+1, a_2-a_3+1; \frac{1}{z}\right)$$
$$+ \left. \frac{\Gamma(a_3)\Gamma(a_1-a_3)\Gamma(a_2-a_3)}{\Gamma(b_1-a_3)\Gamma(b_2-a_3)}(-z)^{-a_3} {}_3F_2\left(a_3, a_3-b_1+1, a_3-b_2+1; -a_1+a_3+1, -a_2+a_3+1; \frac{1}{z}\right)\right\}. \quad (A.14)$$

For integrals with different type of denominator factors, similar calculations bring us to the following results:

$$\int \frac{d\omega}{[\omega^2]^\mu[(k-\omega)^2+m^2]^\nu} = \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(2-\mu-\varepsilon)\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\nu)\Gamma(2-\varepsilon)} {}_2F_1\left(\mu,\mu+\nu+\varepsilon-2; 2-\varepsilon; -\frac{k^2}{m^2}\right), \quad (A.15)$$

$$\int \frac{d\omega}{[\vec{\omega}^2]^\mu[(k-\omega)^2+m^2]^\nu} = \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(\frac{3}{2}-\mu-\varepsilon)\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\nu)\Gamma(3/2-\varepsilon)} {}_2F_1\left(\mu,\mu+\nu+\varepsilon-2; 3/2-\varepsilon; -\frac{\vec{k}^2}{m^2}\right). \quad (A.16)$$

This method can also be applied to integrals with more complicated numerator structure.

For completeness, we also show the first order ε expansion of the integrals arising into the one-loop perturbative expressions considered in this work:

$$\int \frac{d\omega}{(\omega^2+m^2)[(k-\omega)^2+m^2]} = \frac{[m^2]^{-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ \frac{1}{\varepsilon} - \gamma + 2 - \sqrt{1+\frac{4m^2}{k^2}} \ln\left(\frac{\sqrt{1+\frac{4m^2}{k^2}}+1}{\sqrt{1+\frac{4m^2}{k^2}}-1}\right) + \mathcal{O}(\varepsilon) \right\}, \quad (A.17)$$

$$\int \frac{d\omega}{\omega^2[(k-\omega)^2+m^2]} = \frac{[m^2]^{-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ \frac{1}{\varepsilon} - \gamma + 2 - \left(1+\frac{m^2}{k^2}\right)\ln\left(1+\frac{k^2}{m^2}\right) + \mathcal{O}(\varepsilon) \right\}, \quad (A.18)$$

$$\int \frac{d\omega\,\omega_i}{\omega^2[(k-\omega)^2+m^2]} = k_i \frac{[m^2]^{-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ \frac{1}{2}\left(\frac{1}{\varepsilon}-\gamma\right) + 1 + \frac{1}{2}\frac{m^2}{k^2} - \frac{1}{2}\left(1+\frac{m^2}{k^2}\right)^2 \ln\left(1+\frac{k^2}{m^2}\right) + \mathcal{O}(\varepsilon) \right\},$$

$$(A.19)$$

$$\int \frac{đ\omega}{\vec{\omega}^2[(k-\omega)^2+m^2]} = \frac{[m^2]^{-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ \frac{2}{\varepsilon} - 2\gamma + 8 - 2\sqrt{1+\frac{m^2}{\vec{k}^2}} \ln\left(\frac{\sqrt{1+\frac{m^2}{\vec{k}^2}}+1}{\sqrt{1+\frac{m^2}{\vec{k}^2}}-1}\right) + \mathcal{O}(\varepsilon) \right\}. \qquad (A.20)$$

APPENDIX B: CHECKING THE NONSTANDARD INTEGRALS

One way to check analytically the results for the integrals A_m and A_m^4, Eq. (6.36) and Eq. (6.48), respectively, is to make an expansion around $x = 0$ and evaluate the resulting integrals with the help of the Schwinger parametrization. Let us consider then the integral A_m, originally defined in Eq. (6.1). Using Schwinger parameters [25], we can rewrite the denominator factors as exponential functions to give:

$$A_m = \int_0^\infty d\alpha d\beta d\gamma \int đ\omega \exp\left\{-(\alpha+\beta)\omega_4^2 + 2\beta k_4\omega_4 - \beta k_4^2 - (\alpha+\beta+\gamma)\vec{\omega}^2 + 2\beta\vec{k}\cdot\vec{\omega} - \beta\vec{k}^2 - \beta m^2\right\}. \qquad (B.1)$$

Applying similar reasoning as in Ref. [15], we come to the following parametric form of the integral (recall that $x = k_4^2$, $y = \vec{k}^2$):

$$A_m = \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \Gamma(1+\varepsilon) \int_0^1 d\beta \int_0^{1-\beta} \frac{d\alpha}{(\alpha+\beta)^{1/2}} \left[\frac{\alpha\beta}{(\alpha+\beta)}\frac{x}{(x+y+m^2)} + \frac{\beta(1-\beta)y+\beta m^2}{x+y+m^2}\right]^{-1-\varepsilon}. \qquad (B.2)$$

For general values of x, the integral above cannot be solved because of the highly nontrivial denominator factor. Since there can be no singularities at $x = 0$ (this would invalidate the Wick rotation which, as discussed in the text, does hold here), we make an expansion around this point and then integrate. To first order in powers of x we have:

$$A_m \stackrel{x\to 0}{=} \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \Gamma(1+\varepsilon) \int_0^1 d\beta \int_0^{1-\beta} d\alpha\, (\alpha+\beta)^{-1/2} \left\{\left[\beta\frac{m^2+y(1-\beta)}{m^2+y}\right]^{-1-\varepsilon}\right.$$
$$\left. - \left[\beta\frac{m^2+y(1-\beta)}{m^2+y}\right]^{-2-\varepsilon} \left[\frac{\alpha\beta}{(m^2+y)(\alpha+\beta)} - \beta\frac{m^2+y(1-\beta)}{(m^2+y)^2}\right](1+\varepsilon)x + \mathcal{O}(x^2)\right\}. \qquad (B.3)$$

After performing the integration we get:

$$A_m(x,y) \stackrel{x\to 0}{=} \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} (-2) \left\{-\frac{x}{m^2+y} + \frac{1}{\varepsilon}\Gamma(1+\varepsilon){}_2F_1\left(-\varepsilon,2+\varepsilon;1-\varepsilon;-\frac{y}{m^2+y}\right)\right.$$
$$\left. +\sqrt{1+\frac{m^2}{y}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon)\right\}. \qquad (B.4)$$

In the above formula, we isolated the hypergeometric term and evaluate the ε expansion separately. In order to do this, we have to differentiate the hypergeometric function with respect to the parameters. In general, differentiation of ${}_2F_1(a,b;c;z)$ with respect to, e.g. the parameter b, gives (similar expressions are obtained for differentiation with respect to a, c):

$${}_2F_1^{(0,1,0,0)}(a,b;c;z) = \sum_{k=0}^\infty \frac{(a)_k(b)_k \Psi(b+k)}{(c)_k} \frac{z^k}{k!} - \Psi(b){}_2F_1(a,b;c;z), \qquad (B.5)$$

where $\Psi(k)$ is the digamma function and the Pochhammer symbol $(a)_k = \Gamma(a+k)/\Gamma(a)$ (see, for instance, [24]). With the help of formula Eq. (B.5), we get:

$${}_2F_1(-\varepsilon,2+\varepsilon;1-\varepsilon;z) = 1 + \varepsilon\ln(1-z) + \mathcal{O}(\varepsilon). \qquad (B.6)$$

Inserting this back into Eq. (B.4), we can write down the result for the integral A_m (to first order in powers of x):

$$A_m(x,y) \stackrel{x\to 0}{=} \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{-\frac{2}{\varepsilon} + 2\gamma + 2\left[-\ln\left(\frac{m^2}{m^2+y}\right) + \frac{x}{m^2+y}\right]\right.$$
$$\left. -2\sqrt{1+\frac{m^2}{y}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon)\right\}, \qquad (B.7)$$

which agrees explicitly with the corresponding expansion of the result given in Eq. (6.36).

We now turn to the integral A_m^4, given by Eq. (6.2). The parametric form has the expression:

$$A_m^4 = k_4 \frac{(x+y+m^2)^{-1-\varepsilon}\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}} \int_0^1 d\beta \int_0^{1-\beta} d\alpha \, \frac{\beta}{(\alpha+\beta)^{3/2}} \left[\frac{\alpha\beta}{(\alpha+\beta)} \frac{x}{(x+y+m^2)} + \frac{\beta(1-\beta)y+\beta m^2}{x+y+m^2} \right]^{-1-\varepsilon} \quad \text{(B.8)}$$

Calculations similar to the integral A_m bring us to the following result (to first order in x):

$$A_m^4(x,y) \overset{x\to 0}{=} k_4 \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ 2\sqrt{1+\frac{m^2}{y}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1} \right) + 2\left(1+\frac{m^2}{y}\right) \ln\left(\frac{m^2}{m^2+y}\right) \right.$$

$$\left. -\frac{2}{3}\frac{x}{y} \left[1+\left(\frac{m^2}{y}-2\right)\ln\frac{m^2}{m^2+y} - \frac{2}{\sqrt{1+\frac{m^2}{y}}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) \right] + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon) \right\}, \quad \text{(B.9)}$$

again in agreement with the expansion of the result given in Eq. (6.48).

Another useful check comes from the study of the mass differential equation. With I^n given by Eq. (6.3), the derivative with respect to the mass reads:

$$m\frac{\partial I^n}{\partial m} = \int \frac{đ\omega \, \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} \left\{ -2\frac{m^2}{(k-\omega)^2+m^2} \right\}. \quad \text{(B.10)}$$

From the relations Eq. (6.4), Eq. (6.5) and Eq. (B.10) we get the following relation:

$$k_4\frac{\partial I^n}{\partial k_4} + k_k\frac{\partial I^n}{\partial k_k} + m\frac{\partial I^n}{\partial m} = (d+n-5)I^n. \quad \text{(B.11)}$$

Using the same procedures as in the text, we can then derive a differential equation for the integral in terms of the mass:

$$m^2\frac{\partial I^n}{\partial m^2} = (d+n-4)\frac{m^2}{k^2+m^2}I^n - \frac{m^2}{k^2+m^2} \int \frac{đ\omega \, \omega_4^n}{[(k-\omega)^2+m^2]^2\vec{\omega}^2}. \quad \text{(B.12)}$$

Starting with the case $n=0$ where $I^0 \equiv A_m$, we see that by inserting the solution, Eq. (6.36), we have that in the limit $\varepsilon \to 0$

$$m^2\frac{\partial A_m}{\partial m^2} + (1+2\varepsilon)\frac{m^2}{k^2+m^2}A_m = -\frac{m^2(x+y+m^2)^{-2-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{1}{y\sqrt{1+\frac{m^2}{y}}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(\varepsilon). \quad \text{(B.13)}$$

In terms of Schwinger parameters, the explicit integral of Eq. (B.12) reads:

$$-\frac{m^2}{k^2+m^2} \int \frac{đ\omega}{[(k-\omega)^2+m^2]^2\vec{\omega}^2} = -\frac{m^2}{x+y+m^2} \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}} \int_0^1 d\alpha \, (1-\alpha)^{-1/2-\varepsilon}(m^2+\alpha y)^{-1-\varepsilon} \quad \text{(B.14)}$$

and for $m^2 \neq 0$ indeed

$$-\frac{m^2}{k^2+m^2} \int \frac{đ\omega}{[(k-\omega)^2+m^2]^2\vec{\omega}^2} = -\frac{m^2(x+y+m^2)^{-2-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{1}{y\sqrt{1+\frac{m^2}{y}}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(\varepsilon), \quad \text{(B.15)}$$

showing that the mass differential equation is satisfied. For $m^2 = 0$, the right-hand side of Eq. (B.13) vanishes as $m^2 \ln m^2$, whereas the parametric form of the integral in Eq. (B.14) goes like m^2/ε. The integral of Eq. (B.14) does contain an ambiguity in the ordering of the limits $m^2 \to 0$ and $\varepsilon \to 0$, but this problem is not of importance because of the overall factor m^2 in the differential equation. In fact, since the solution of the mass differential equation is in principle formally derived as the integral over m^2 and $m^2 = 0$ is the only the limit of this integral, the ambiguity encountered may be regarded as an integrable singularity and presents no problem.

Turning now to the case $n = 1$ where $I^1 \equiv A_m^4$, we first extract the overall k_4 factor as before by defining $A^4 = k_4 \overline{A}_m$ such that the differential equation is

$$m^2 \frac{\partial \overline{A}_m}{\partial m^2} = -2\varepsilon \frac{m^2}{k^2 + m^2} \overline{A}_m - \frac{m^2}{k^2 + m^2} \int \frac{d\!\!^-\omega}{[(k-\omega)^2 + m^2]^2 \vec{\omega}^2}. \tag{B.16}$$

Notice that in the integral term we have used the identities

$$\int \frac{d\!\!^-\omega\, \omega_4}{[(k-\omega)^2 + m^2]^2 \vec{\omega}^2} = \int \frac{d\!\!^-\omega\, (k_4 - \omega_4)}{[\omega^2 + m^2]^2 \left(\vec{k} - \vec{\omega}\right)^2} = k_4 \int \frac{d\!\!^-\omega}{[(k-\omega)^2 + m^2]^2 \vec{\omega}^2}. \tag{B.17}$$

Now, for $m^2 \neq 0$, the integral term of Eq. (B.16) is finite as $\varepsilon \to 0$; however, the $m^2 = 0$ limit is again ambiguous but as above this can be regarded as an integrable singularity. Also, when $m^2 = 0$, $\overline{A}$ is known to be ε finite (it is the massless integral considered in Ref. [15]). This means that as $\varepsilon \to 0$ we have the simple integral expression

$$m^2 \frac{\partial \overline{A}_m}{\partial m^2} = -\frac{1}{x + y + m^2} \frac{1}{(4\pi)^{2-\varepsilon}} \frac{1}{y\sqrt{1 + \frac{m^2}{y}}} \ln \left(\frac{\sqrt{1 + \frac{m^2}{y}} + 1}{\sqrt{1 + \frac{m^2}{y}} - 1} \right). \tag{B.18}$$

Knowing the solution, Eq. (6.48), it suffices to show that when $m^2 = 0$ the original massless integral from Ref. [15] is reproduced and that the derivative of the massive solution satisfies the above. Both of these steps are straightforward.

---

[1] E. S. Abers and B. W. Lee, Phys. Rept. **9**, 1 (1973).
[2] V. N. Gribov, Nucl. Phys. B **139** (1978) 1.
[3] D. Zwanziger, Nucl. Phys. B **485**, 185 (1997) [arXiv:hep-th/9603203].
[4] D. Zwanziger, Nucl. Phys. B **518** (1998) 237.
[5] G. Burgio, M. Quandt and H. Reinhardt, arXiv:0807.3291 [hep-lat].
[6] N. H. Christ and T. D. Lee, Phys. Rev. D **22**, 939 (1980) [Phys. Scripta **23**, 970 (1981)].
[7] A. P. Szczepaniak and E. S. Swanson, Phys. Rev. D **65** (2002) 025012 [arXiv:hep-ph/0107078].
[8] A. P. Szczepaniak, Phys. Rev. D **69** (2004) 074031 [arXiv:hep-ph/0306030].
[9] C. Feuchter and H. Reinhardt, Phys. Rev. D **70** (2004) 105021 [arXiv:hep-th/0408236]; C. Feuchter and H. Reinhardt, arXiv:hep-th/0402106.
[10] H. Reinhardt and C. Feuchter, Phys. Rev. D **71** (2005) 105002 [arXiv:hep-th/0408237].
[11] H. Reinhardt and D. Epple, Phys. Rev. D **76**, 065015 (2007) [arXiv:0706.0175 [hep-th]].
[12] H. Reinhardt, Phys. Rev. Lett. **101**, 061602 (2008) [arXiv:0803.0504 [hep-th]].
[13] D. R. Campagnari and H. Reinhardt, Phys. Rev. D **78**, 085001 (2008) [arXiv:0807.1195 [hep-th]].
[14] P. Watson and H. Reinhardt, Phys. Rev. D **75**, 045021 (2007) [arXiv:hep-th/0612114].
[15] P. Watson and H. Reinhardt, Phys. Rev. D **76** (2007) 125016 [arXiv:0709.0140 [hep-th]].
[16] P. Watson and H. Reinhardt, Phys. Rev. D **77** (2008) 025030 [arXiv:0709.3963 [hep-th]].
[17] H. Reinhardt and P. Watson, arXiv:0808.2436 [hep-th].
[18] P. Watson and H. Reinhardt, arXiv:0711.2997 [hep-th].
[19] M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions", Dover Publications, New York, 1970.
[20] L. Lewin, "Polylogarithms and associated functions", Elsevier North Holland, 1981.
[21] T. Muta, "Foundations of Quantum Chromodynamics", World Scientific, Singapore, 1987.
[22] C. S. Fischer, J. Phys. G **32**, R253 (2006) [arXiv:hep-ph/0605173].
[23] E. E. Boos and A. I. Davydychev, Theor. Math. Phys. **89** (1991) 1052 [Teor. Mat. Fiz. **89** (1991) 56].
[24] H. Bateman, "Higher Transcendental Functions", McGraw-Hill Book company, New York, 1953.
[25] P. Pascual, R. Tarrach, "QCD: Renormalization for the Practitioner", Springer Verlag, Berlin, 1984.